CENTER FOR THE APPLICATION OF MATHEMATICS

Lehigh University
Bethlehem, Pa.

DISTORTION OF A FINITE AMPLITUDE STANDING WAVE IN AN ELASTIC PANEL

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Technical Report No. CAM-110-6 August 1969
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Distortion of a finite amplitude standing wave in an elastic panel.

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Abstract:

The equations governing the stretching of a nonlinear elastic panel of finite length are integrated in the limit when the strains are small but the strain rates are comparable with the natural frequency of the panel. It is shown that such a theory must be used when investigating the distortion and decay of a disturbance which is produced by impact and reflection. The mathematical problem involves functions whose values repeat at a sequence of times which are separated by a time interval which depends on the current value of the function.

In a future paper we show that resonant vibrations can also be treated by this theory.

1. INTRODUCTION

Most of the results which have been obtained to describe nonlinear effects in wavelike disturbances are valid in the limit when the disturbance is generated by a single progressing wave. These results are applicable when the transmitting medium can be regarded as of semi-infinite extent so that the effects of reflections from the boundaries, which excite additional components of the disturbance, may be neglected. In this and future reports we investigate a class of small amplitude phenomena whose occurrence depends on the nonlinear interaction, either in the body of the transmitting medium or at its boundaries, between different components of a disturbance.

Only those one dimensional disturbances which are governed by systems of totally hyperbolic equations which are of the non-dispersive type are considered, so that the equation of state of the transmitting medium is rate independent. When only one component of a disturbance in such a system is excited it is generated by the passage of a simple wave. Moreover, according to classical linear theory <u>any</u> disturbance can be thought of as generated by the passage of nondistorting, nonattenuated simple (progressing) waves which, in general, interact at the boundaries but not in the body of the material. The mathematical problem consists of determining the signals carried by these waves from prescribed initial and boundary data. It usually reduces to solving linear difference equations.

The small amplitude disturbances which we investigate can roughly be divided into three classes. In class I are those disturbances for which, to a first approximation, there is no interaction or distortion of the component waves in the body of the material but, nevertheless, for which nonlinear effects are important at the boundaries. That is, although nonlinearity is of secondary importance when determining how the component waves propagate it is of primary importance when determining the signals they carry. An example of such a situation occurs when a column of gas in a Kundt tube is driven at a resonant frequency (see Chester [1963]).

Class II disturbances are those for which even though the components do not interact in the body of the medium, nonlinearity (amplitude dispersion) is not only important at the boundaries but also may distort the signal carried by the component waves so that shocks may form and ultimately dissipate the disturbance. Such phenomena are usually associated with the cummulative effect of locally small nonlinearity. A typical example of such a disturbance occurs during the starting problem when an elastic panel, initially at rest, which is rigidly bonded at one end is deformed by a periodically applied traction at the other. According to the theory of linear elasticity, which neglects all dissipative mechanisms, the resulting motion at any particle is the sum of an oscillation at the natural frequency of the panel plus an oscillation at the frequency of the applied traction. In practice, however, the standing wave component is attenuated. In a future paper, we show that if all dissipative mechanisms are neglected everywhere but in shock layers, the disturbance which occurs is actually of class II and that our theory for such disturbances does agree with observation, even at resonance.

Finally, class III disturbances are those for which one component is affected by its interaction with the other components which do not themselves interact. An example of this is the Melde phenomena (see Rayleigh [1945]). There, a transverse vibration is controlled by its interaction with a lower amplitude stretching wave. The frequency of the transverse vibration is a subharmonic of that of the applied tension. A perfectly analagous situation occurs in magneto-hydrodynamics where the transverse magnetic field is controlled by the acoustic wave. These disturbances are of obvious importance in problems of control.

In this paper, as an example of a class II disturbance, we describe the evolution of a small amplitude stretching disturbance in an elastic panel which has one plane boundary rigidly bonded when the other parallel boundary, whose normal motion generated the disturbance, is brought to rest. The disturbance is described in terms of the variations in strain and particle velocity in the panel at some reference time t=0 when the boundaries are at rest. Other motions which are mathematically equivalent are that produced in an inviscid gas when the tube in which the gas is in motion is closed at both ends, and that produced in a vibrating string after both ends are held fixed. According to classical linear theory, if all dissipative mechanisms are neglected, the resulting motion is oscillatory with a period determined by the ambient sound speed a_0 and width L of the transmitting medium. In practice the motion is not exactly periodic and, because no energy is being fed into the system at the boundaries, dissipative mechanisms will ultimately attenuate the disturbance. Which of the many possible dissipative mechanisms is mainly responsible for the decay will depend upon the particular system as well as upon the details of the disturbance at t=0. In part I of this report we describe how nonlinearity (amplitude dispersion) produces large strain rates and ultimately shocks. In part II we describe how these shocks attenuate the disturbance and produce a uniformly strained panel.

When the disturbance in the panel is produced by an impulsive motion of one boundary the initial disturbance can be calculated. It is generated by the passage of a simple wave (G.I. Taylor [1941]). In section 2 we use this disturbance to show that if M denotes the ratio of the second to first order elastic constants and if e(t,X) denotes the strain then, the linear theory is valid only if the amplitude of the disturbance is small in the sense that

$$|Me| << 1 \tag{1.1}$$

and the strain rate is small in the sense that

$$\left|M\frac{\partial e}{\partial t}\right| << \frac{a_o}{L} . \tag{1.2}$$

In the limit when (1.1) holds but when (1.2) need not the signal distorts and may generate shocks. In this small amplitude finite rate limit, e is not, in general, obtained as an explicit function of (X,t), as in the linear theory: both e and t are given as explicit functions of X and some parameter α, which is invariant at a characteristic wavelet. Motivated by the representation of the solution in a simple wave, in section 3 we derive a representation of the most general disturbance in an elastic panel in the small amplitude limit without restricting the level of the strain rate. According to this representation the disturbance is produced by the passage of two distorting simple waves which only interact at boundaries.

In section 4 we show that the finite rate theory must be used to discuss the evolution of the disturbance in an elastic panel when its boundaries are held fixed. Nonlinear difference equations are obtained for the signals carried by the component waves. These are solved exactly and a complete representation is obtained in terms of conditions in the panel at t=0. As an application of the theory, the deformation which results when a boundary is moved in an arbitrary manner over a time less than $2a_0/L$, and then is held at rest, is calculated in terms of the prescribed boundary motion of the panel. In figures 4 and 5 we illustrate how a disturbance which at t=0 corresponds to a sinusoidal variation in strain and zero particle velocity evolves to form shocks. In part II we show how these shocks attenuate the disturbance. We also generalize our results to disturbances which are governed by general systems of hyperbolic equations.

2. LIMITATIONS OF LINEAR THEORY

We consider one-dimensional longitudinal disturbance in an elastic material of finite reference length L. Let

$$x = x(X,t) \tag{2.1}$$

be the position at time t, measured from one boundary X=0 of the material, of the material particle "X" which in some reference state, when the material was at constant density ρ_0 and constant hydrostatic pressure p_0, was at a distance X from X=0. Let $T(X,t)$ be the traction measured from this reference state. Then, the momentum equation for the material relates T to the material velocity

$$u(X,t) = x_{,t} \tag{2.2}$$

by

$$T_{,X} = \rho_o u_{,t}. \tag{2.3}$$

For isentropic deformations of an elastic material

$$T = T(e) \tag{2.4}$$

is a known function of the strain

$$e = x_{,X} - 1. \tag{2.5}$$

In particular, for small strains we assume that

$$T = E[e+Me^2+0(e^3)]. \tag{2.6}$$

A special case of an elastic material is an inviscid polytropic gas which, for isentropic flows, has an equation of state

$$p = p_o\left(\frac{\rho}{\rho_o}\right)^\gamma \tag{2.7}$$

where the density ρ is related to e by

$$\rho(1+e) = \rho_o. \tag{2.8}$$

According to (2.7) and (2.8), for an inviscid gas

$$T = p_o - p = \gamma p_o\left[e - \frac{\gamma+1}{2}e^2 + 0(e^3)\right], \tag{2.9}$$

so that

$$E = \gamma p_o \quad \text{and} \quad M = -\frac{\gamma+1}{2}. \tag{2.10}$$

For algebraic convenience we choose L as the unit of length and L/a_o as the unit of time where

$$a_o = \sqrt{\frac{E}{\rho_o}} \tag{2.11}$$

is the sound speed in the reference state. We also take E as the unit of stress and a_o as the unit of speed. With this scaling, (2.3) reads

$$a^2(e)e,_X = u,_t \tag{2.12}$$

where

$$a^2 = T'(e) = 1+2Me+0(e^2). \tag{2.13}$$

Equations (2.12) and (2.13) are supplemented by the continuity equation

$$u,_X = e,_t \tag{2.14}$$

which is obtained by eliminating x(X,t) from (2.2) and (2.5). Riemann (see Stoker [1957]) showed that if

$$c(e) = \int_0^e a(s)ds, = e\{1+\tfrac{1}{2}Me+0(e^2)\}, \tag{2.15}$$

then, in any isentropic wave motion governed by (2.12) and (2.14), the combination of variables

$$f = \tfrac{1}{2}[u-c(e)] \tag{2.16}$$

is invariant at each α-wavelet which propagates so that

$$\left.\frac{DX}{Dt}\right|_\alpha = a(e), \tag{2.17}$$

while the combination of variables

$$g = \frac{1}{2}[u+c(e)] \qquad (2.18)$$

is invariant at each β-wavelet which propagates so that

$$\frac{DX}{Dt}\Big|_{\beta} = -a(e). \qquad (2.19)$$

If

$$f = F(t) \text{ at } X = 0$$

while $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ (2.20)

$$g = G(t) \text{ at } X = 1,$$

and if an α-wavelet is tagged by the time if left X=0, while a β-wavelet is tagged by the time if left X=1, then according to (2.16), (2.18) and (2.20)

$$u = G(\beta)+F(\alpha), \quad \text{and} \quad c = G(\beta)-F(\alpha). \qquad (2.21)$$

Classical linear theory formally appro. mates a(e) by unity in (2.17) and (2.19). Then, (2.21) imply that

$$u = G(\beta_L)+F(\alpha_L), \quad \text{and} \quad c = G(\beta_L)-F(\alpha_L) \qquad (2.22)$$

where F and G are functions of the forward, and backward linear characteristic variables

$$\alpha_L = t-X, \quad \text{and} \quad \beta_L = t+(X-1). \qquad (2.23)$$

The statements (2.22) and (2.23) imply that, according to the formal linear theory, the most general disturbance can be regarded as the superposition of two components which do not interact in the body of the material: a non-distorting, non-attenuated, progressing wave moving to the right (the α-wave), and a non-distorting progressing wave moving to the left, (the β-wave). These components only interact at the boundaries of the material and the mathematical problem reduces to determining the signal functions F and G from prescribed boundary and initial data. This usually involves solving linear difference equations.

Since

$$a(e) = 1+Me+O(e^2), \tag{2.24}$$

it might be thought that the linear theory provides a good approximation in the small strain limit when

$$|Me|<<1. \tag{2.25}$$

Then, to a good approximation, the traction T(e) and c(e) can be calculated from the strain according to the linear laws

$$T = e, \quad \text{and} \quad c = e. \tag{2.26}$$

However, as is well known, the small amplitude assumption (2.25) is not, by itself, sufficient to justify the use of linear theory to determine the variation in $e(X,t)$: the strain rates must also be "small". The most relevant illustration of this limitation, which is important in what follows, occurs in the signalling, or starting, problem. Suppose that the elastic panel is rigidly bonded at X=1 while the other boundary X=0 is at rest for all times except in the interval $-\varepsilon \leq t \leq 0$ when it is moved in a prescribed way. If no shocks form, the pulse which is generated at X=0 over this interval moves towards the boundary X=1 as a simple wave, (see G.I. Taylor [1941]). For at any X, $G(\beta)\equiv 0$ for times $t<t_a(X)$ - the arrival time of the wavelet $\beta=1-\varepsilon$ which left X=1 at the arrival of the front of the pulse. Then, according to (2.21)

$$u = F(\alpha) \quad \text{and} \quad c = -F(\alpha), \tag{2.27}$$

and (2.17) integrates to give

$$t = \alpha + X/a(e) \tag{2.28}$$

for the arrival time at X of the characteristic wavelet α which left X=0 at t=α. The wave described by (2.27) and (2.28) is amplitude dispersed: constant levels of u and c are carried by characteristic wavelets each of which moves with an invariant speed which is determined by the value of c it carries. Linear theory predicts that in the simple wave

where

$$\left.\begin{aligned} c &= -F(\alpha_L), = c_L \text{ say}, \\ t &= \alpha_L + X. \end{aligned}\right\} \tag{2.29}$$

Conditions (2.27), (2.28) and (2.29), together with the mean value theorem of differential calculus, imply that

$$\begin{aligned} \left|\frac{c_L}{c} - 1\right| &= \left|\frac{F(\alpha_L)-F(\alpha)}{c}\right| \\ &= \left|\frac{(\alpha_L-\alpha)}{c}F'(\theta)\right|, \text{ for some } \theta \text{ between } \alpha \text{ and } \alpha_L, \\ &= D(c)X|F'(\theta)|, \end{aligned} \tag{2.30}$$

where

$$D(c) = \left|\frac{1-a(c)}{ac}\right| = |M|[1+0(c)]. \tag{2.31}$$

Equations (2.25), (2.26), (2.30) and (2.31) imply that the linear theory, which neglects amplitude dispersion, yields a good approximation to conditions in a simple wave only if the amplitude of the signal function is small in the sense that

$$|MF(t)| \ll 1, \tag{2.32}$$

and if the rate of change of the signal function is small in the sense that

$$|MF'(t)| \ll 1. \tag{2.33}$$

Condition (2.32) states that the amplitude of imposed strain must be small compared with the magnitude of the ratio of first to second order elastic constants, (= $\frac{2}{\gamma+1}$ for a gas). Condition (2.33) states that the amplitude of the product of travel time of the pulse front and the imposed strain rate must also be small compared with the magnitude of the ratio of first to second order elastic constants. Theories for which the restriction (2.32) holds but the restriction (2.33) does not are called small amplitude, finite rate theories. Such theories were first used to solve significant problems in gasdynamics by Whitham (1956). They have been generalized, and used to solve problems governed by general systems of hyperbolic equations by Varley and his co-workers (1965), (1967) and (1969). Up to now these theories have only been used to investigate disturbances which are generated by progressing waves, as in the signaling problem. In this report we show that such theories can be generalized to account for effects due to reflected waves. As an application, we show that such a theory is necessary when investigating the decay of a standing wave, or the "shut-off" problem in an elastic panel. In future reports we will show that resonant wave motions can also be analyzed by the theory.

For future reference, and to motivate the mathematical approach we use to generalize the theory, we briefly review conditions in a small amplitude but finite rate simple wave. To a first approximation, according to (2.26) and (2.27)

$$c = e = -u = -F(\alpha) \tag{2.34}$$

where the arrival time $t=t(\alpha,X)$ of the characteristic wavelet

α is given by (2.28), and (2.24), as

$$t-X = \alpha+MF(\alpha)X. \qquad (2.35)$$

Because the α-wave described by (2.34) and (2.35) is amplitude dispersed, its profile distorts as it propagates. A measure of this distortion is the <u>incremental</u> <u>arrival</u> <u>time</u> of the α-wavelet at X,

$$p(\alpha,X) = t,_{\alpha} = \alpha_{L,\alpha} = 1+MF'(\alpha)X. \qquad (2.36)$$

At any α-wavelet, the ratio of the strain rate to input strain rate

$$e,_t/e,_t|_{X=0} = e,_X/e,_X|_{X=0} = p^{-1}. \qquad (2.37)$$

According to linear theory $p\equiv 1$ and the level of the strain rate in the material is always bounded by its value at input. This theory gives a valid approximation when

$$X<<|MF'(\alpha)|^{-1}. \qquad (2.38)$$

By contrast the finite rate theory predicts that if $|F'(\alpha)|>|M|^{-1}$ at some wavelet then, before the α-wave reaches X=1, $p\to 0$ at some (X,t) and the profile distorts so much that the strain rate becomes unbounded compared with its level at input and at least one weak shock forms. <u>The main effect of such shocks is to attenuate the disturbance</u>. We will show in part II that in an elastic material, where all attenuating mechanisms are neglected everywhere but in shock layers, it is this shock attenuation which is actually responsible for the decay of a standing wave.

Shocks attenuate a simple wave because the characteristic wavelets, each of which carries a constant value of e and u, coalesce into it. In fact, according to the elastic model, if a shock is allowed to propagate into an "infinite region" all wavelets except those carrying vanishingly small values of u and e will coalesce into it, or some other shock, so that ultimately the disturbance will be fully attenuated. The equations governing conditions at a shock in an elastic material are well known (see Varley [1967]). Because they are needed in what follows, we briefly indicate how they are derived. If $t=W(X)$ denotes the arrival time of the shock at X, if $\alpha^+(X)$ and $\alpha^-(X)$ are the characteristic wavelets immediately ahead and behind the shock at X, then the usual jump conditions at a weak shock imply that

$$W'(X) = 1 + \tfrac{1}{2}M\{F(\alpha^+)+F(\alpha^-)\} \qquad (2.39)$$

while the condition that α^+ and α^- are at the shock at the same time, together with the statement (2.34), also imply that

$$W-X = \alpha^+ + MF(\alpha^+)X = \alpha^- + MF(\alpha^-)X. \qquad (2.40)$$

Once the input signal $F(\alpha)$ is specified, equations (2.39) and (2.40) govern the variations in $W(X)$, $\alpha^+(X)$ and $\alpha^-(X)$ at any shock. For the special case of a shock moving into an undisturbed region, where $F(\alpha^+)\equiv 0$ for $\alpha^+ \leq -\varepsilon$, these equations integrate to give

$$X = -\frac{2}{MF^2(\alpha^-)}\int_{-\varepsilon}^{\alpha^-} F(\alpha)\,d\alpha, \quad \text{and} \quad t = X+\alpha^- + MF(\alpha^-)X \qquad (2.41)$$

as the shock trajectory. According to (2.41), only those wavelets at which $MF<0$ can coalesce into this shock. If $\alpha=0$ is the next zero of $F(\alpha)$ after $\alpha=-\varepsilon$, then the amplitude of the pulse generated at $X=0$ over the time interval $-\varepsilon \leq t \leq 0$ is vanishingly small at distances which are large compared with the attenuation length,

$$\ell_a = -2M\int_{-\varepsilon}^{0} F(s)ds, \qquad (2.42)$$

of the pulse. For, according to (2.41), as $X/\ell_a \to \infty$ the amplitude of this pulse, and the shock,

$$-MF(\alpha^-) \sim (\ell_a/X)^{1/2}. \qquad (2.43)$$

Note that the decay in strength of the pulse as it propagates is determined by ℓ_a which is a measure of the total energy content of the input signal.

3. SMALL AMPLITUDE, FINITE RATE THEORY.

In this section we show that when both the signal functions F and G are not identically zero, the linear approximation, (2.22) and (2.23), is valid only when their amplitudes are small in the sense that both

$$|MF(t)| \ll 1, \quad \text{and} \quad |MG(t)| \ll 1 \qquad (3.1)$$

while their rates of change are small in the sense that both

$$|MF'(t)| \ll 1, \quad \text{and} \quad |MG'(t)| \ll 1. \qquad (3.2)$$

In (3.2) a dash denotes differentiation with respect to time t, whose scale of measurement has been chosen so that the travel time of a sonic front from boundary to boundary of the material is unity. We integrate equations (2.12) and (2.14) in the small amplitude finite rate limit when the restrictions (3.1) hold but when (3.2) need not. In contrast to the linear theory, the finite rate theory predicts that amplitude dispersion distorts, and that weak shocks attenuate the disturbance. However, the two theories have one important feature in common: the α-wave and β-wave do not interact in the body of the material. It is this one single feature which makes the standing wave problem, as well as that of resonant motions, mathematically tractable. Even though, in general, the problem of determining the signal functions F and G from the prescribed initial and boundary data involves solving nonlinear difference equations, it can be solved for these problems to show that the assumptions (3.2), inherent in the linear approximation, are invalid and that the finite rate theory must be used.

Since the equations governing any disturbance are linear, with known coefficients, if the hodograph variables (f,g), and (t,X), are used as independent, and dependent, variables, this choice of variables might seem the natural one. However, the boundary value problem associated with the class of disturbances considered here, which requires that

$$\text{when } X = 0, \quad t = F^{-1}(f)$$

while

$$\text{when } X = 1, \quad t = G^{-1}(g)$$

is a formidable free boundary value problem in terms of these variables even when the signal functions F(t) and G(t) are known.† The hodograph description also has two other disadvantages. It is difficult to interpret any physical approximation in terms of these variables. For example, the disturbance generated by a simple wave, which is characteristic of the disturbances studied here, cannot be described in a straightforward way in the hodograph formalism. The second disadvantage is that the use of the hodograph technique is limited to disturbances which are governed by a system of two first order hyperbolic equations.

The procedure used here, which can be extended to general systems of hyperbolic equations, is to choose variables which are optimal for the particular component of the disturbance which is being investigated. These variables are different for the α-wave component and

†Ludford [1952] tackled this formidable problem. He used the hodograph approach to calculate the time at which shocks form.

β-wave component of the disturbance. This approach works because, to a first approximation, these components do not interact in the body of the material.

3.1 The α-wave.

Motivated by the representation (2.34) and (2.35), to describe conditions in the α-wave we use (α,X) as basic independent variables. As basic dependent variables we take the incremental arrival time

$$p(\alpha,X) = t,_{\alpha}(\alpha,X), \tag{3.3}$$

and $c(\alpha,X)$. In terms of these variables

$$u = c+2F(\alpha). \tag{3.4}$$

Differentiating (2.17) with respect to α at constant X, and considering $a(c)$ as a function of c rather than e, yields

$$p,_X + \frac{a'(c)}{a^2(c)} c,_{\alpha} = 0. \tag{3.5}$$

Differentiating the last of equations (2.21) with respect to α at constant β yields

$$\frac{Dc}{D\alpha}\Big|_{\beta} = c,_{\alpha} + \frac{DX}{D\alpha}\Big|_{\beta} c,_X = -F'(\alpha). \tag{3.6}$$

Since

$$\frac{Dt}{D\alpha}\Big|_{\beta} = \begin{cases} p+a^{-1} \dfrac{DX}{D\alpha}\Big|_{\beta} & \text{from (2.17)}, \\ -a^{-1} \dfrac{DX}{D\alpha}\Big|_{\beta} & \text{from (2.19)}, \end{cases} \tag{3.7}$$

$$\frac{DX}{D\alpha}\Big|_{\beta} = -\frac{a}{2} p. \tag{3.8}$$

If (3.8) and (3.6) are used to eliminate $c,_{\alpha}$, condition (3.5) yields

$$(\sqrt{a(c)}\ p),_X = [a(c)]^{-\frac{3}{2}}a'(c)F'(\alpha), \qquad (3.9)$$

which is regarded as an equation governing the variation of p at each α-wavelet. The characteristic condition (3.9) is supplemented by (3.6) which, using (3.8), reads

$$\left.\frac{Dc}{D\alpha}\right|_{\beta} = c,_{\alpha} - \frac{1}{2}a(c)pc,_X = -F'(\alpha). \qquad (3.10)$$

Equations (3.9) and (3.10), which are two first order equations for $p(\alpha,X)$ and $c(\alpha,X)$, are in a form which is specially suited for investigating conditions in an α-wave and, in particular, how it is affected by its interaction with the β-wave. Suppose, for example, we wish to determine conditions in the α-wave which is generated at X=0 for times $t\geq 0$. Conditions in this wave are, in theory, determined by the values of $F(\alpha)$ and $c(\alpha,1)$ at values of $\alpha\geq 0$ together with the value of $c(0,X)$ for $0\leq X\leq 1$, (see fig. 1). We say that the interaction of the β-wave with the α-wave can be neglected if the trajectory of any wavelet $\alpha=\alpha_1$ is determined by $F(\alpha)$ for $0\leq\alpha\leq\alpha_1$ and is independent of $c(\alpha,1)$ and $c(0,X)$. There is no such interaction in the small amplitude, finite rate limit. For, without restricting $F'(\alpha)$, integrating (3.9) from X=0 and using the mean value theorem of integral calculus together with the restriction $|Mc|<<1$ to estimate the error yields

$$p = 1+MF'(\alpha)X \qquad (3.11)$$

as a uniformly valid first approximation for the incremental arrival time $p=t,_{\alpha}$. Equation (3.9) and the mean value theorem also implies that (3.11) can be formally integrated to give

$$t = X+\alpha+MF(\alpha)X \tag{3.12}$$

which is a uniformly valid implicit equation for the first approximation to $\alpha(t,X)$. According to (3.12), just as in a simple wave, the trajectory of any α-wavelet depends only on the signal $F(\alpha)$ it carries.

To obtain higher order approximations for conditions in the α-wave the iteration scheme

$$(\sqrt{a(c_n)}\ p_n)_{,X} = [a(c_n)]^{-\frac{3}{2}}a'(c_n)F'(\alpha) \tag{3.13}$$

and

$$c_{n+1,\alpha}-\frac{1}{2}a(c_n)p_n c_{n+1,X} = -F'(\alpha), \tag{3.14}$$

with

$$c_o = 0, \quad \text{and} \quad p_o = 1+MF'(\alpha)X \tag{3.15}$$

may be used. Higher order approximations may also be obtained as the limiting case as $|MF(\alpha)|\to 0$ of the regular perturbation scheme in which the Riemann invariant

$$r = c+F(\alpha) = \varepsilon[r_o(\alpha,X)+\varepsilon r_1(\alpha,X)+0(\varepsilon^2)] \tag{3.16}$$

and

$$p = p_o(\alpha,x)+\varepsilon p_1(\alpha,X)+0(\varepsilon^2) \tag{3.17}$$

where

$$\varepsilon = \max.|r|.$$

Since in a simple wave $r\equiv 0$, (3.16) and (3.17) describe a perturbed simple wave. Note that the perturbation is only regular if r and p are regarded as functions of X and the exact characteristic variable α.

When an analysis similar to that described by equations (3.5)-(3.12) is applied to the β-component of the disturbance using (β,X) as independent, and (t,c) as dependent, variables, a complete representation of conditions in any small amplitude, finite rate disturbance is obtained. It is given by

$$u = G(\beta)+F(\alpha) \tag{3.18}$$

and

$$c = G(\beta)-F(\alpha), \tag{3.19}$$

where $\alpha(t,X)$ and $\beta(t,X)$ are given <u>implicitly</u> by

$$t-X = \alpha+MF(\alpha)X \tag{3.20}$$

and

$$t+(X-1) = \beta+MG(\alpha)(X-1). \tag{3.21}$$

If $\alpha(t,X)$ is computed from (3.20) then, in the limit when $MF(\alpha)\to 0$ but $MF'(\alpha)$ is unrestricted, $F(\alpha(t,X))$ is a uniformly valid first approximation for F as a function of (t,X). Moreover, to obtain the first approximation to $\frac{\partial t}{\partial X}$ and $\frac{\partial t}{\partial \alpha}$ in this limit, equation (3.20) may be formally differentiated to give

$$\lim_{MF\to 0} \frac{\partial t}{\partial X} = \lim_{MF\to 0} 1+MF(\alpha) = 1$$

and

$$\lim_{MF\to 0} \frac{\partial t}{\partial \alpha} = \lim_{MF\to 0} [1+MF'(\alpha)X]$$
$$= 1+MF'(\alpha)X.$$

To a <u>second</u> approximation, of course,

$$\frac{\partial t}{\partial X} = 1-Mc = 1+M[F(\alpha)-G(\beta)].$$

The linear theory formally approximates α and β in (3.20) and (3.21) by the <u>explicit</u> expressions

$$\alpha_L = t-X \tag{3.22}$$

and

$$\beta_L = t+(X-1) \tag{3.23}$$

This is permissable if and only if conditions (3.1) and (3.2) are satisfied.

4. DISTORTION OF A STANDING WAVE

In this section we use the representation (3.18)-(3.21) to describe the disturbance in an elastic panel after its boundaries X=0 and X=1 are brought to rest. This disturbance is described for all $t \geq 0$ in terms of the variations in $\phi(X)$ and $\psi(X)$, $0 \leq X \leq 1$, the values of F and G in the panel at t=0. These are related to u and c in the panel at t=0 by (3.18) and (3.19). The nonlinear mixed initial and boundary value problem which is solved here reduces to determining the signal functions $F(\alpha)$ and $G(\beta)$ from (3.20) and (3.21) in the region $0 \leq X \leq 1$ for all $t \geq 0$ from the initial data that when t=0,

$$F = \phi(X) \text{ and } G = \psi(X), \quad 0 \leq X \leq 1, \qquad (4.1)$$

and from the boundary data that on X=0 and X=1

$$u = G(\beta)+F(\alpha) = 0, \quad \text{all } t \geq 0. \qquad (4.2)$$

For comparison, and to motivate the steps we take to solve the nonlinear problem, we review the argument used to construct the solution to the linear problem. Reference to fig. 2 will be helpful.

4.1 Linear theory

Linear theory, according to (3.22) and (3.23), predicts that in the region $0 \leq X \leq 1$ when $t \geq 0$ the characteristic variables (α,β) range from -1 to ∞. To determine the signal functions $F(\alpha)$ and $G(\beta)$ over this range, note that the condition that (4.2) holds on X=0 when, by (3.23),

$$\beta = \alpha-1, \quad \alpha \geq 0, \quad \beta \geq -1 \qquad (4.3)$$

implies that F and G are related by the condition that

$$F(R)+G(R-1) = 0 \quad \text{when } R \geq 0. \qquad (4.4)$$

Similarly, the condition that (4.2) holds on X=1 when, by (3.22),

$$\alpha = \beta - 1 \tag{4.5}$$

implies that

$$F(S-1)+G(S) = 0 \text{ when } S\geq 0. \tag{4.6}$$

If G is eliminated, by taking

$$R = S+1, \tag{4.7}$$

(4.4) and (4.6) imply that

$$F(S+1) = F(S-1) \text{ when } S\geq 0, \tag{4.8}$$

or, equivalently, that

$$F(\alpha+2) = F(\alpha) \text{ when } \alpha\geq -1. \tag{4.9}$$

Condition (4.9) states that $F(\alpha)$ is periodic with period 2 for all $\alpha\geq -1$: if

$$F(\alpha) = f(\alpha) \text{ when } -1\leq\alpha<1 \tag{4.10}$$

then

$$F(\alpha) = f(\alpha-2m) \text{ when } 2m-1\leq\alpha<2m+1, \quad m=0,1,2,\ldots \tag{4.11}$$

Similarly, $G(\beta)$ is periodic with period 2 for all $\beta\geq -1$ so that if

$$G(\beta) = g(\beta) \text{ when } -1\leq\beta<1 \tag{4.12}$$

then

$$G(\beta) = g(\beta-2n) \text{ when } 2n-1\leq\beta<2n+1, \quad n=0,1,2,\ldots \tag{4.13}$$

According to (3.18), (3.19), (3.22), (3.23), (4.11) and (4.13) one representation of the solution, which as we will see is the one most easily generalized to the nonlinear case, is that

$$u = g(s)+f(r), \quad \text{and } c = g(s)-f(r), \quad -1\leq s,r<1, \tag{4.14}$$

where $r(t,X)$ and $s(t,X)$ are given explicitly by the conditions that

$$t-(X+2m) = r \text{ when } 2m-1\leq t-X<2m+1, \quad m=0,1,2,.. \tag{4.15}$$

while

$$t+(X-1-2n) = s \text{ when } 2n-1\leq t+(X-1)<2n+1, \quad n=0,1,2,.. \tag{4.16}$$

To determine f and g in terms of ϕ and ψ note that conditions (4.15) and (4.16) imply that when $t=0$

$$m = n = 0, \quad \text{and} \quad X = -r = s+1. \tag{4.17}$$

Since,

$$f = \phi(X) \text{ and } g = \psi(X) \text{ when } t=0, \tag{4.18}$$

$$f(r) = \phi(-r) \text{ when } -1\leq r\leq 0, \tag{4.19}$$

and

$$g(s) = \psi(1+s) \text{ when } -1\leq s\leq 0. \tag{4.20}$$

To determine $f(r)$ over the range $0\leq r<1$ and $g(s)$ over the range $0\leq s<1$ we use conditions (4.4) and (4.6). These yield

$$f(r) = -g(r-1) = -\psi(r) \text{ when } 0\leq r\leq 1, \tag{4.21}$$

and

$$g(s) = -f(s-1) = -\phi(1-s) \text{ when } 0 \le s \le 1. \qquad (4.22)$$

In (4.17)-(4.22) we have assumed that the boundaries $X=0$ and $X=1$ are not suddenly brought to rest at $t=0$ so that u is single valued and equal to zero at $(X,t) = (0,0)$ and $(X,t) = (1,0)$.

The statements (4.13)-(4.16) with f and g determined from ϕ and ψ by (4.19)-(4.22) represents a complete solution, according to linear theory, of the posed mixed initial and boundary value problem. Note that, according to (4.11) and (4.13), the linear theory predicts that if the rates of change of $F(\alpha)$ and $G(\beta)$ are small, in the sense (3.2), over the range $-1 \le \alpha, \beta \le 1$ then they remain small in this sense for all t. Now, even though the linear theory is consistent in this sense, it is, nevertheless, wrong.† As we show in what follows, even if the rates of change of $F(\alpha)$ and $G(\beta)$ over the range $-1 \le \alpha, \beta \le 1$ are small in the sense (3.2), ultimately, for sufficiently large time, conditions (3.2) are always violated and the finite rate theory must be used.

4.2 Finite rate theory.

Conditions (3.20) and (3.21) imply that in the finite rate theory the ranges of the characteristic variables (α, β) are (see fig. 3)

$$-\tfrac{1}{2}P_0 \le \alpha \le \infty \text{ and } -\tfrac{1}{2}Q_0 \le \beta \le \infty, \qquad (4.23)$$

where

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†We mean by this that the linear theory of elasticity makes a prediction which is in disagreement with the non-linear theory of elasticity.

$$P_0 = 2[1-MG(0)] \text{ and } Q_0 = 2[1+MF(0)]. \tag{4.24}$$

To determine the signal functions $F(\alpha)$ and $G(\beta)$ over this range we use an argument which is similar to that used for the linear theory. The condition that (4.2) holds on $X=0$ for all $t = \alpha>0$ when, by (3.21),

$$\beta = \alpha-1+MG(\beta), = \alpha-1-MF(\alpha) \tag{4.25}$$

implies that

$$F(R)+G(R-1-MF(R)) = 0, \quad \text{all } R\geq 0. \tag{4.26}$$

Similarly, the condition that (4.10) also holds on $X=1$ for all $t = \beta\geq 0$ when, by (3.20),

$$\alpha = \beta-1-MF(\alpha), = \beta-1+MG(\beta) \tag{4.27}$$

implies that

$$F(S-1+MG(S))+G(S) = 0, \quad \text{all } S\geq 0. \tag{4.28}$$

To obtain a single equation for $F(\alpha)$ in the range $\alpha\geq-\frac{1}{2}P_0$ we take

$$R = 1+MF(R)+S, = 1-MG(S)+S, \quad S\geq 0, \quad R\geq\frac{1}{2}P_0 \tag{4.29}$$

in (4.26) to give

$$F(S+1-MG(S))+G(S) = 0, \quad S\geq 0. \tag{4.30}$$

Conditions (4.28) and (4.30) state that if

$$\alpha = S-[1-MG(S)], \quad S\geq 0, \quad \alpha\geq-\frac{1}{2}P_0 \tag{4.31}$$

then

$$F(\alpha)+G(S) = 0 \tag{4.32}$$

and

$$F(\alpha+2[1-MG(S)])+G(S) = 0, \tag{4.33}$$

or, eliminating G, that

$$F(\alpha+P(\alpha)) = F(\alpha) \tag{4.34}$$

when

$$P(\alpha) = 2[1+MF(\alpha)], \quad \text{all } \alpha \geq -\tfrac{1}{2}P_o. \tag{4.35}$$

If, by a procedure similar to that outlined above, F is eliminated, equations (4.26) and (4.28) yield the equations

$$G(\beta+Q(\beta)) = G(\beta) \tag{4.36}$$

when

$$Q(\beta) = 2[1-MG(\beta)], \quad \text{all } \beta \geq -\tfrac{1}{2}Q_o, \tag{4.37}$$

for the variation in G.

Equation (4.34), with P given in terms of F by (4.35), is a nonlinear functional equation for F. It states that even though F is not periodic with fixed period 2 it is invariant, or repeats, at a sequence of times which are separated by an invariant time interval P=2(1+MF). In particular, the zeros of F repeat at times which are separated by a time interval 2. When F(β) is continuous, the solution to (4.34) can be expressed in terms of its variations over the range $-\frac{1}{2}P_o \leq \beta \leq \frac{1}{2}P_o$ where, according to (4.24), (4.31), (4.32) and (4.35)

$$P_o = P(-\tfrac{1}{2}P_o) \tag{4.39}$$

If

$$F(\alpha) = f(\alpha) \text{ when } -\tfrac{1}{2}P_o \le \alpha \le \tfrac{1}{2}P_o \tag{4.40}$$

then the solution to (4.34) is given parametrically by

$$F(\alpha) = f(r), \quad -\tfrac{1}{2}P_o \le r \le \tfrac{1}{2}P_o \tag{4.41}$$

where, when

$$(m-\tfrac{1}{2})P_o \le \alpha \le (m+\tfrac{1}{2})P_o, \quad m=0,1,2,\ldots, \tag{4.42}$$

$r(\alpha)$ is given implicitly by

$$\alpha = r+2m(1+Mf(r)). \tag{4.43}$$

To check that (4.40)-(4.43) does indeed represent the solution of (4.34) and (4.35), let α_m and α_{m+1} by the values of α in the ranges $(m-\tfrac{1}{2})P_o \le \alpha_m \le (m+\tfrac{1}{2})P_o$ and $(m+\tfrac{1}{2})P_o \le \alpha_{m+1} \le (m+\tfrac{3}{2})P_o$ which are the images under the mapping defined by (4.42) and (4.43) of the same parameter r so that, by (4.40),

$$F(\alpha_{m+1}) = F(\alpha_m) = f(r). \tag{4.44}$$

Then, since

$$\left.\begin{aligned} \alpha_m &= r+2m(1+Mf(r)) \\ \text{and} \quad \alpha_{m+1} &= r+2(m+1)(1+Mf(r)) \end{aligned}\right\}, \tag{4.45}$$

$$\alpha_{m+1}-\alpha_m = 2(1+Mf(r)) = P(\alpha_m) = P(\alpha_{m+1}) \tag{4.46}$$

Equations (4.44) and (4.46) state that F is equal at two values of α whose difference in P. This agrees with the statements (4.34) and (4.35).

In contrast to the linear theory, the solution represented by (4.40)-(4.43) predicts that even though the level of

$|MF(\alpha)|$ is bounded by its level over the range $-\frac{1}{2}P_o \le \alpha \le \frac{1}{2}P_o$, the level of $|MF'(\alpha)|$ is not bounded by its level over this range. In fact, at some $\alpha=\alpha_c$, which is an image under the mapping (4.43) of some point $r=r_c$ at which $Mf'(r_c)<0$, $|MF'(\alpha_c)|$ is unbounded. This implies that a shock forms at $X=0$ at time $t=\alpha_c$, or somewhere else in the panel before time t_c. (How the analysis must be modified once shocks form is discussed in part II.) To see this note that, by (4.41) and (4.43),

$$F'(\alpha) = f'(r)\frac{dr}{d\alpha} = f'(r)[1+2mMf'(r)]^{-1}. \qquad (4.47)$$

Equation (4.47) implies that $|MF'(\alpha)|$ first becomes unbounded, compared with $|Mf'(r)|$, at some time

$$t_c = \alpha_c = r_c+2m_c(1+Mf(r_c)), \qquad (4.48)$$

where m_c is the least integer which satisfies the condition that

$$-2Mf'(r_c) = m_c^{-1} \qquad (4.49)$$

for some r_c in the range $-\frac{1}{2}P_o \le r_c \le \frac{1}{2}P_o$. Since, as we will show when we determine $f(r)$ in terms of conditions in the panel at $t=0$,

$$f(-\tfrac{1}{2}P_o) = f(\tfrac{1}{2}P_o), \qquad (4.50)$$

there is always some r_c which satisfies (4.49) for some $m_c>0$.

The signal function $G(\beta)$ can be determined from conditions (4.36) and (4.37) by an argument similar to that used to determine $F(\alpha)$. If

$$G(\beta) = g(\beta) \text{ when } -\tfrac{1}{2}Q_o \le \beta \le \tfrac{1}{2}Q_o, \qquad (4.51)$$

then

$$G(\beta) = g(s), \quad -\tfrac{1}{2}Q_o \le s \le \tfrac{1}{2}Q_o \tag{4.52}$$

where, when

$$(n-\tfrac{1}{2})Q_o \le \beta \le (n+\tfrac{1}{2})Q_o, \quad n=0,1,2,\ldots \; ; \tag{4.53}$$

$s(\beta)$ is given implicitly by

$$\beta = s+2n(1-Mg(s)) \tag{4.54}$$

If the statements (4.41)-(4.43) and (4.52)-(4.54) are used to determine $F(\alpha)$ and $G(\beta)$, (3.18)-(3.21) can be used to represent the disturbance in the panel in terms of f and g. It is given by

$$u = g(s)+f(r), \quad \text{and} \quad c = g(s)-f(r), \tag{4.55}$$

where

$$-\tfrac{1}{2}P_o \le r \le \tfrac{1}{2}P_o, \quad \text{and} \quad -\tfrac{1}{2}Q_o \le s \le \tfrac{1}{2}Q_o: \tag{4.56}$$

in the range

$$(m-\tfrac{1}{2})P_o \le t-\tfrac{1}{2}P_o X \le (m+\tfrac{1}{2})P_o, \quad m=0,1,2,\ldots , \tag{4.57}$$

$r(t,X)$ is given implicitly by

$$t-(X+2m) = r+Mf(r)(X+2m); \tag{4.58}$$

in the range

$$(n-\tfrac{1}{2})Q_o \le t+\tfrac{1}{2}Q_o(X-1) \le (n+\tfrac{1}{2})Q_o, \quad n=0,1,2\ldots \tag{4.59}$$

$s(t,X)$ is given implicitly by

$$t+(X-1-2n) = s+Mg(s)(X-1-2n). \tag{4.60}$$

4.3 Determination of f and g in terms of conditions in the panel at t=0

If, when t=0,

$$f = \phi(X) \text{ and } g = \psi(X), \quad 0\le X\le 1, \tag{4.61}$$

then f(r) is given parametrically by

$$f(r) = \begin{cases} \phi(-R) \text{ when } r = R[1+M\phi(-R)], & -1\le R\le 0, \ -\frac{1}{2}P_o\le r\le 0; \\ -\psi(R) \text{ when } r = R[1-M\psi(R)], & 0\le R\le 1, \ 0\le r\le\frac{1}{2}P_o: \end{cases} \tag{4.62}$$

g(s) is given by

$$g(s) = \begin{cases} \psi(1+S) \text{ when } s = S[1-M\psi(1+S)], & -1\le S\le 0, \ -\frac{1}{2}Q_o\le s\le 0; \\ -\phi(1-S) \text{ when } s = S[1+M\phi(1-S)], & 0\le S\le 1, \ 0\le s\le\frac{1}{2}Q_o. \end{cases} \tag{4.63}$$

In (4.61)-(4.63) f and g are continuous because it is assumed that the panel is not suddenly brought to rest at t=0 so that

$$\psi(o)+\phi(o) = 0, \quad \text{and} \quad \psi(1)+\phi(1) = 0. \tag{4.64}$$

The results of linear theory, described in section (4.1), are recovered by formally taking M=0 in (4.55)-(4.64).

The arguments used to establish the results (4.62) and (4.63) parallel those already used in section (4.1) for the linear theory. To obtain the first part of (4.62) first note that, according to (4.57) and (4.58), when t=0 and $0\le X\le 1$

$$m = 0 \text{ and } r = -X[1+Mf(r)]. \tag{4.65}$$

However, by (4.61), when t=0

$$f(r) = \phi(X), \quad 0\le X\le 1, \tag{4.66}$$

which, together with (4.65), imply the first part of (4.62).

Similarly, to obtain the first part of (4.63) note that (4.59) and (4.60) imply that when $t=0$ and $0\le X\le 1$,

$$n = 0 \text{ and } s = (X-1)[1-Mg(s)]. \qquad (4.67)$$

However, by (4.61), when $t=0$

$$g(s) = \psi(X) \qquad (4.68)$$

which, together with (4.67), imply the first part of (4.63).

It remains to establish the second parts of (4.62) and (4.63). To determine $f(r)$ in the range $0\le r\le \frac{1}{2}P_o$ we use (4.26). This relates f at any argument in this range to $g(s)$ at a negative argument where it is given by the first part of (4.63). In fact, according to (4.26)

$$f(r) = -g(s) \qquad (4.69)$$

when

$$r = s+1-Mg(s), \quad 0\le r\le \tfrac{1}{2}P_o, \quad -\tfrac{1}{2}Q_o\le s\le 0. \qquad (4.70)$$

Equations (4.69) and (4.70) together with the representation (4.63) for $g(s)$ in the range $-\frac{1}{2}Q_o\le s\le 0$ yields the second part of (4.62). Finally, to determine $g(s)$ in the range $0\le s\le \frac{1}{2}Q_o$ we use (4.28) and the first part of (4.62).

The statements (4.55)-(4.60) with $f(r)$ and $g(s)$ given by (4.62) and (4.63) provide a complete <u>algebraic</u> description of conditions in any small amplitude finite rate disturbance in terms of quite general conditions in the panel at $t=0$. In section 6 we show that these statements are still valid away from weak shocks whose influence is to attenuate the disturbance.

4.4 Determination of f and g for the impact problem.

A problem of practical interest is the determination of the motion produced in an elastic panel, or bar, which is rigidly bonded at X=1 after the other end, X=0, is suddenly moved and then held fixed. The analagous problem in gas dynamics is the determination of the flow in a gas filled tube, which is closed at one end, when a piston is suddenly moved and then brought to rest at the other. Before the front of the disturbance reaches the boundary X=1, it is generated by the simple wave which was described in section 2. If the duration of the motion, ε, of the boundary X=0 is short in the sense that

$$\varepsilon<2, \tag{4.71}$$

that is, if $u\not\equiv 0$ at X=0 only over a time interval which is less than that taken by a sound front to leave and return to X=0 after reflection from X=1, the signal functions f(r) and g(s) can readily be expressed in terms of the motion of the boundary X=0.

If at X=0

$$u = U(t), \tag{4.72}$$

where

$$U(-\varepsilon) = U(o) = 0 \tag{4.73}$$

and where

$$U\equiv 0 \text{ outside the range } -\varepsilon<t<0 \tag{4.74}$$

then, for $-\varepsilon\leq t<1-\varepsilon$

$$F(\alpha) = U(\alpha) \text{ and } G(\beta) \equiv 0 \tag{4.75}$$

so, by (3.18)-(3.21), the disturbance is generated by a simple wave. For $t\geq -1$ ($\leq 1-\varepsilon$) the disturbance is described by (4.55)-(4.60) with

$$f(r) = \begin{cases} U(r), \text{ for } -\frac{1}{2}P_o \le r \le 0, \\ U(\sigma-2) \text{ when } r = \sigma+2MU(\sigma-2),\ 0\le\sigma\le\sigma_o,\ 0\le r\le\frac{1}{2}P_o, \end{cases} \tag{4.76}$$

where

$$\tfrac{1}{2}P_o+MU(-\tfrac{1}{2}P_o) = 1 \quad \text{and} \quad \sigma_o = \tfrac{3}{2}P_o-2; \tag{4.77}$$

with

$$g(s) = -U(\tau-1), \text{ when } s = \tau+MU(\tau-1),\ -1\le\tau\le 1,\ -1\le s\le 1. \tag{4.78}$$

The statements (4.76)-(4.78) are readily established by using arguments similar to those used in section (4.3).

5. DISCUSSION AND APPLICATIONS

In section 3 we pointed out that the linear approximation to F as a function of (t,X) is valid only if $|MF(t)|$ and $|MF'(t)|$ are small compared with unity. However, as for the standing wave problem discussed in section 4, in most applications F is not known *a priori* but must be calculated as a solution to some difference equation. An obvious question which arises is what are the restrictions which must be imposed on F which allow the difference equation to be replaced by the difference equation which is predicted by linear theory. There is no reason to assume that they are the same as those which allow F(·) to be approximated by F(t-X): the restrictions will depend on the particular problem being investigated. To illustrate this point, we derive the conditions which allow the solution to the nonlinear difference equation (4.34) and (4.35), which is given by (4.41)-(4.43), to be approximated by the solution to the linear difference equation (4.9).

For convenience we take $P_0=2$, (this is always possible by a suitable choice of reference configuration). Then, (4.40) and (4.43) imply that

$$F(t) = f(r), \quad -1 \leq r \leq 1, \tag{5.1}$$

where r(t) is given by

$$t = r+2m(1+Mf(r)). \tag{5.2}$$

The linear theory predicts that

$$F_L(t) = f(r_L), \tag{5.3}$$

where

$$r_L = t-2m = r+2mMf(r). \tag{5.4}$$

According to (5.1)-(5.4), and the mean value theorem of differential calculus,

$$\left|\frac{F_L(t)}{F(t)} - 1\right| = \left|\frac{f(r_L)-f(r)}{f(r)}\right|, = |2mMf'(\theta)|, \quad (5.5)$$

for some θ between r and r_L. According to (5.5) only when

$$\max.|2mMf'(\theta)|<<1 \quad (5.6)$$

does the solution $F_L(t)$ to the linear difference equation approximate the solution F(t) to the nonlinear difference equation. Since

$$MF'(t) = Mf'(r)/1+2mMf'(r), \quad (5.7)$$

for all times at which (5.6) is satisfied the small rate assumption $|MF'(t)|<<1$ is also satisfied, so that $F(\alpha)$ can be approximated by F(t-X). However, the converse is not true. Even though $|MF'(t)|<<1$ over some time interval the restriction (5.6) need not hold: even though the distortion of the signal carried by the α-wavelets may be neglected for some time, (i.e. $F(\alpha) = F(t-X)$), nevertheless, it does not follow that amplitude dispersion (nonlinearity) may also be neglected in calculating the signal carried. According to (5.6) and (5.7) a restriction on MF'(t) which allows the solution of the nonlinear difference equation to be approximated by the solution to the linear difference equation is that

$$\left|\frac{MF'(t)}{(2m)^{-1}-MF'(t)}\right|<<1. \quad (5.8)$$

Chester [1963] has shown that small amplitude resonant vibrations can also be investigated by a theory in which amplitude dispersion can be neglected in the body of the transmitting material but not in the calculation of the signal which is carried, (which contains shocks!) In a future paper we show, however, that the more general finite rate theory must be used to investigate conditions in the resonant band and also to investigate the approach to resonance.

We have not shown that the solution to the nonlinear difference equation (4.34) and (4.35) does, in fact, itself approximate the solution to the exact equation satisfied by F in the limit when $MF \to 0$. Presumably, this question could be resolved by using the hodograph representation of the solution. However, in partial support of our assumption that it does, we note that there is an equation of state for which the difference equation can be solved exactly so that the question can be answered. This is the equation of state for which the coefficient of $F'(\alpha)$ in (3.9)

$$[a(c)]^{-\frac{3}{2}}a'(c) \equiv \text{constant}, = M \text{ say.}^{\dagger} \tag{5.9}$$

By a suitable choice of E and M this equation of state can be used to locally approximate many other equations of state when investigating small amplitude phenomena which depend on the magnitudes of both the local tangent and curvature of the stress/strain curve. When (5.9) is used, equation (3.9) integrates to give

$$\frac{\partial t}{\partial \alpha} = (1-\tfrac{1}{2}Mc)\{\bar{\alpha}(\alpha)+MXF'(\alpha)\}. \tag{5.10}$$

The function $\bar{\alpha}(\alpha)$ is determined as a function of $F(\alpha)$ from the conditions that

$$\text{on } X = 0: \quad t = \alpha, \quad u = 0 \quad \text{and} \quad c = -2F(\alpha): \tag{5.11}$$

it is given by

†The equation of state (5.9) is an excellent approximation to the actual stress/strain curves of a wide variety of "locking materials" over the full range of strain. It is currently being used by Cekirge and Varley to investigate large amplitude waves in such materials.

$$\bar{\alpha} = [1+MF(\alpha)]^{-1}. \tag{5.12}$$

Similarly,

$$\frac{\partial t}{\partial \beta} = (1-\tfrac{1}{2}Mc)\{\bar{\beta}(\beta)+M(X-1)G'(\beta)\} \tag{5.13}$$

where $\bar{\beta}$ is determined as a function of $G(\beta)$ from the condition that

$$\text{on } X = 1: \quad t = \beta, \quad u = 0 \quad \text{and} \quad c = 2G(\beta): \tag{5.14}$$

it is given by

$$\bar{\beta} = [1-MG(\beta)]^{-1}. \tag{5.15}$$

If now (5.10) is integrated at X=1 and if the condition that

$$\text{on } X = 1, \quad c = 2G(\beta) = -2F(\alpha) \tag{5.16}$$

is used we obtain the condition that

$$\text{on } X = 1, \quad t = \alpha+t_1+MF(\alpha)[1+\tfrac{1}{2}MF(\alpha)], \tag{5.17}$$

for some constant t_1. Similarly, (5.13) and the conditions that

$$\text{on } X = 0: \quad c = -2F(\alpha) = 2G(\beta) \tag{5.18}$$

implies that

$$\text{on } X = 0: \quad t = \beta+t_2-MG(\beta)[1+\tfrac{1}{2}MG(\beta)] \tag{5.19}$$

when t_2 is a constant. If now the three conditions (5.11), (5.18) and (5.19) on X=0 are used, together with the three conditions (5.14), (5.16) and (5.17) on X=1, then, by using arguments which are similar to those used in section 4.2, a nonlinear difference equation for F is obtained. Quite remarkably, if the strain is measured so that $t_1+t_2=2$, <u>the exact equation for F is identical to that given for F by (4.34) and (4.35). That is, equations (4.34) and (4.35), which we hope are good approximations for any equation of state in the limit when $MF\to 0$, are exact for the equation of state (5.9) for any level of MF.</u>

5.1 <u>Associated pure initial value problem</u>

One prediction of linear theory is that the solution to the mixed initial and homogeneous boundary value problem, which is given in section 4.1, can also be derived as part of the solution to a pure initial value problem in the region $-\infty\leq X\leq\infty$. The initial values of F and G for this associated problem are periodic functions of X (all X) with fixed period 2 and with

$$F = f(-X) \quad \text{for} \quad -1\leq X\leq 1, \tag{5.20}$$

and with

$$G = g(X-1) \quad \text{for} \quad 0\leq X\leq 2, \tag{5.21}$$

where f and g are given by (4.62) and (4.63) with M=0. The form of the dependence of f and g on u and c imply that the initial values of u and c for this associated problem are also periodic in X and, in addition, that u is odd, while c is even, with respect to both points X=0 and X=1.

According to the representation (4.55)-(4.60) the solution to the nonlinear mixed initial-boundary value problem can also be constructed from the solution to a nonlinear pure initial value problem. Moreover, the initial values, $\Phi(X)$ and $\Psi(X)$, of F and G for the associated problem are obtained from their prescribed values, $\phi(X)$ and $\psi(X)$, over the range $0\leq X\leq 1$ by the identical procedure used in linear theory.† For, (4.58) states that

$$\Phi(X) = f(r), \quad -\tfrac{1}{2}P_o \le r \le \tfrac{1}{2}P_o, \quad -\infty \le X \le \infty \tag{5.22}$$

where r(X) is given by the condition that

$$X+2m = -\frac{r}{1+Mf(r)} \text{ when } -2m-1 \le X \le 1-2m, \quad m = ..-1,0,1,.. \tag{5.23}$$

According to (5.22) Φ is invariant, or repeats, at values of X which are the images under the mapping (5.23) of the same parameter r but different values of m. Any two neighboring points of this sequence are at a fixed distance of 2 units (twice the width of the panel) apart irrespective of the value of r so that

$$\Phi(X+2) = \Phi(X). \tag{5.24}$$

In a similar fashion, it can be shown that at any later time F is still periodic in X with fixed period 2. It follows that even though the solution to the associated initial value problem is not periodic in time, as in the linear theory, it is periodic in X.

5.2 Distortion of a normal mode.

Another representation of any solution to the linear problem is as a sum of its noninteracting, nondistorting, Fourier modes. For a nonlinear system, however, the separate Fourier components into which any initial disturbance can be decomposed do interact so that in the finite rate theory this representation is invalid. In fact, even a pure initial sinusoidal disturbance will distort in time, generating higher harmonics, until shocks form . These ultimately attenuate the disturbance. In part II of this report we show that the ultimate equilibrium state of the material is a state of rest in which the strain is uniform and is given by

$$e_M = \int_0^1 e(t,X)dX. \tag{5.25}$$

(e_M is independent of time because the width of a panel which is held fixed at both boundaries does not change.) This final configuration, in which the shocks have smeared out all spatial inhomogeneities, is an appropriate reference configuration from which the strain and stored elastic energy should be measured. With this convention

$$e_M = 0. \tag{5.26}$$

This reference configuration is attained as the asymptotic limit of a sequence of deformation for which, between shocks, essentially

$$\frac{\partial u}{\partial t} \equiv \frac{\partial e}{\partial X} \equiv 0 \tag{5.27}$$

while

$$\frac{\partial e}{\partial t} = \frac{\partial u}{\partial X} = \text{constant}. \tag{5.28}$$

Asymptotically, for "large" time, at any fixed time, the profile of u against X is a series of N-waves and that of e against X a series of square waves; at any fixed X, however, the profile of u against t is a series of square waves and that of e against t a series of N-waves.

In figures 4 and 5, as an illustration of the theory, we show how the disturbance corresponding to the initial conditions

$$u=0 \quad \text{and} \quad Me = \frac{1}{100}(\sin\pi X - \frac{2}{\pi}) \tag{5.29}$$

(e is measured to have zero mean) develops. We graph the variations of Mu and Me as functions of X at times which correspond to 8 and 14 vibrations according to linear theory.

At these times linear theory predicts u and e are again given by (5.29). Note how the profiles are approaching the asymptotic profiles given by (5.27) and (5.18) and that there is a tendency to equipartion energy between kinetic and potential energy. This will be discussed in greater detail in part II.

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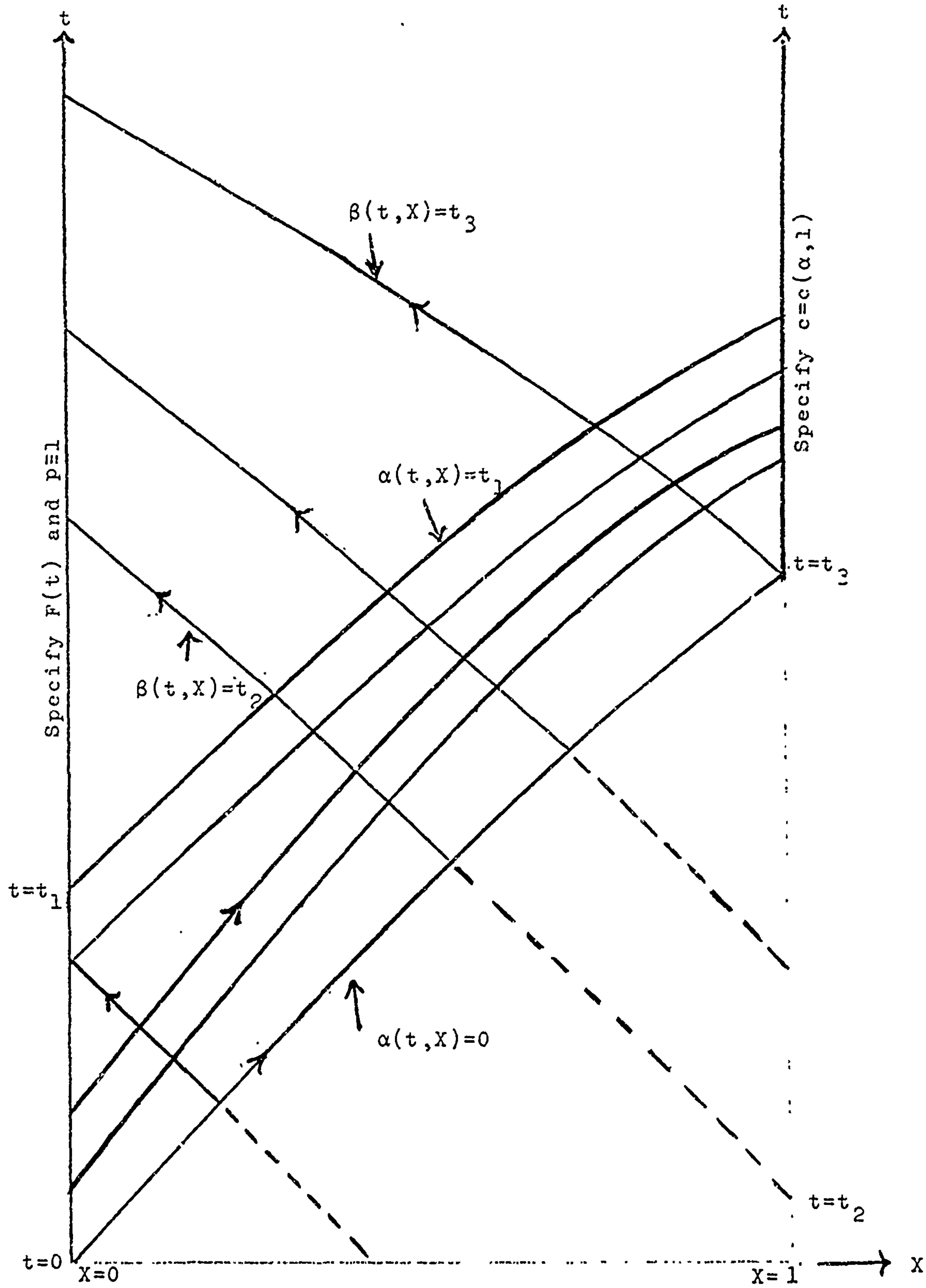

FIGURE 1. $c(\alpha,X)$ and $p(\alpha,X)$ determined by equations (3.9) and (3.10) in region spanned by full curves if $F(\alpha)$ $c(0,X)$ and $c(1,X)$ are known.

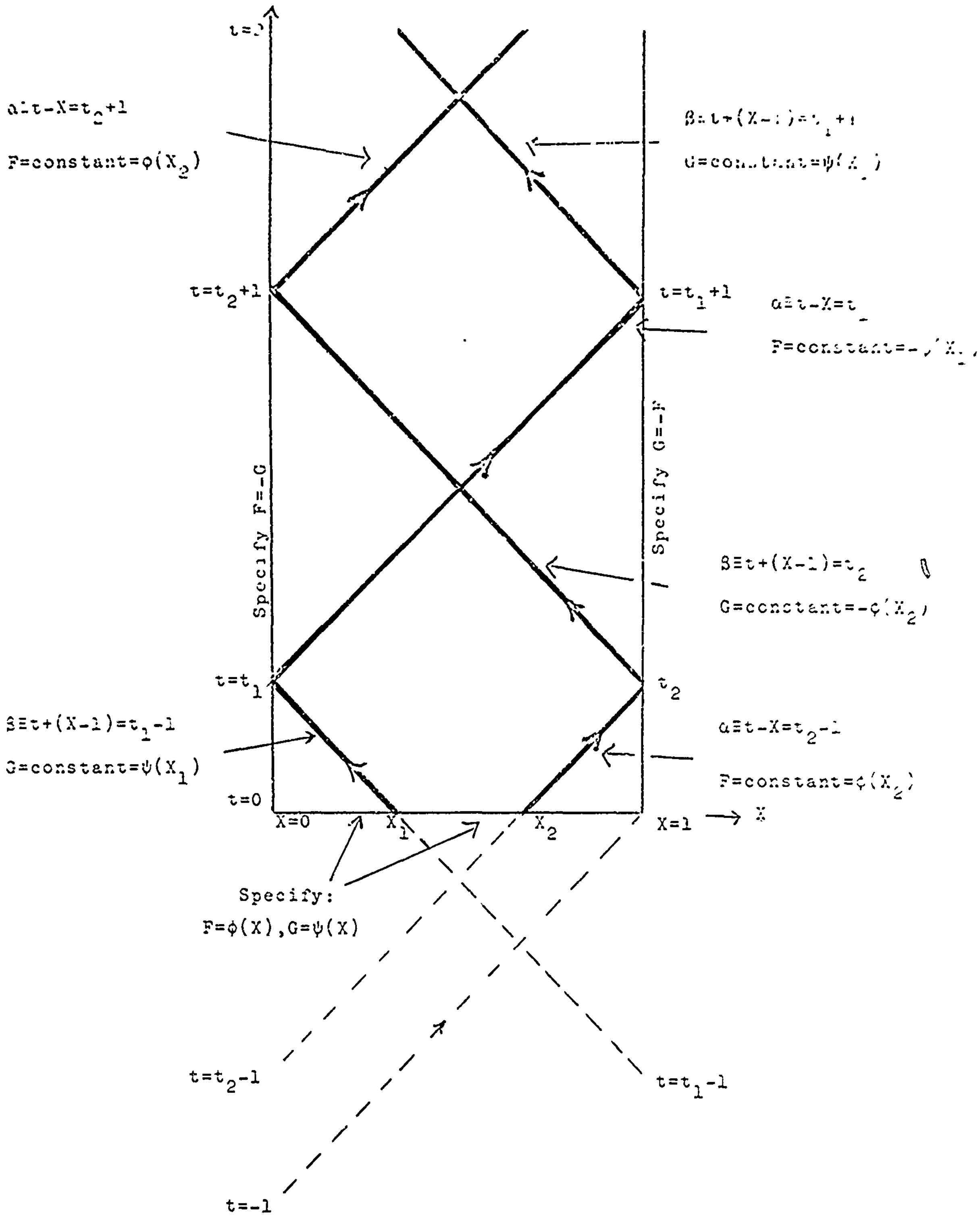

FIGURE 2. Determinition of conditions in panel over a time interval 2 in terms of conditions in panel at t=0 according to linear theory. Pattern repeats in time with period 2.

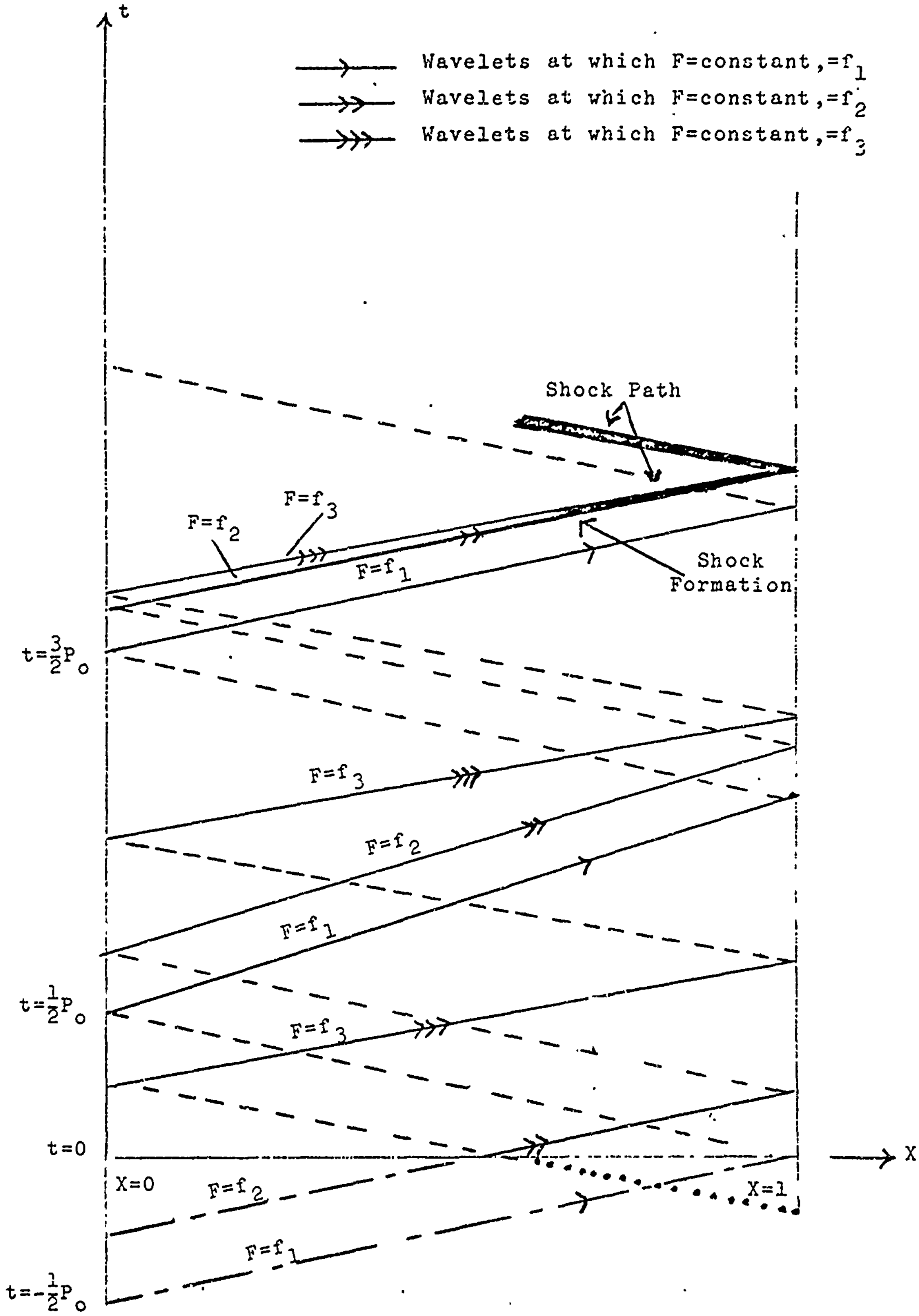

FIGURE 3. Showing how α-wavelets which carry constant values of F (and reflected β-wavelets) coalesce to form a shock.

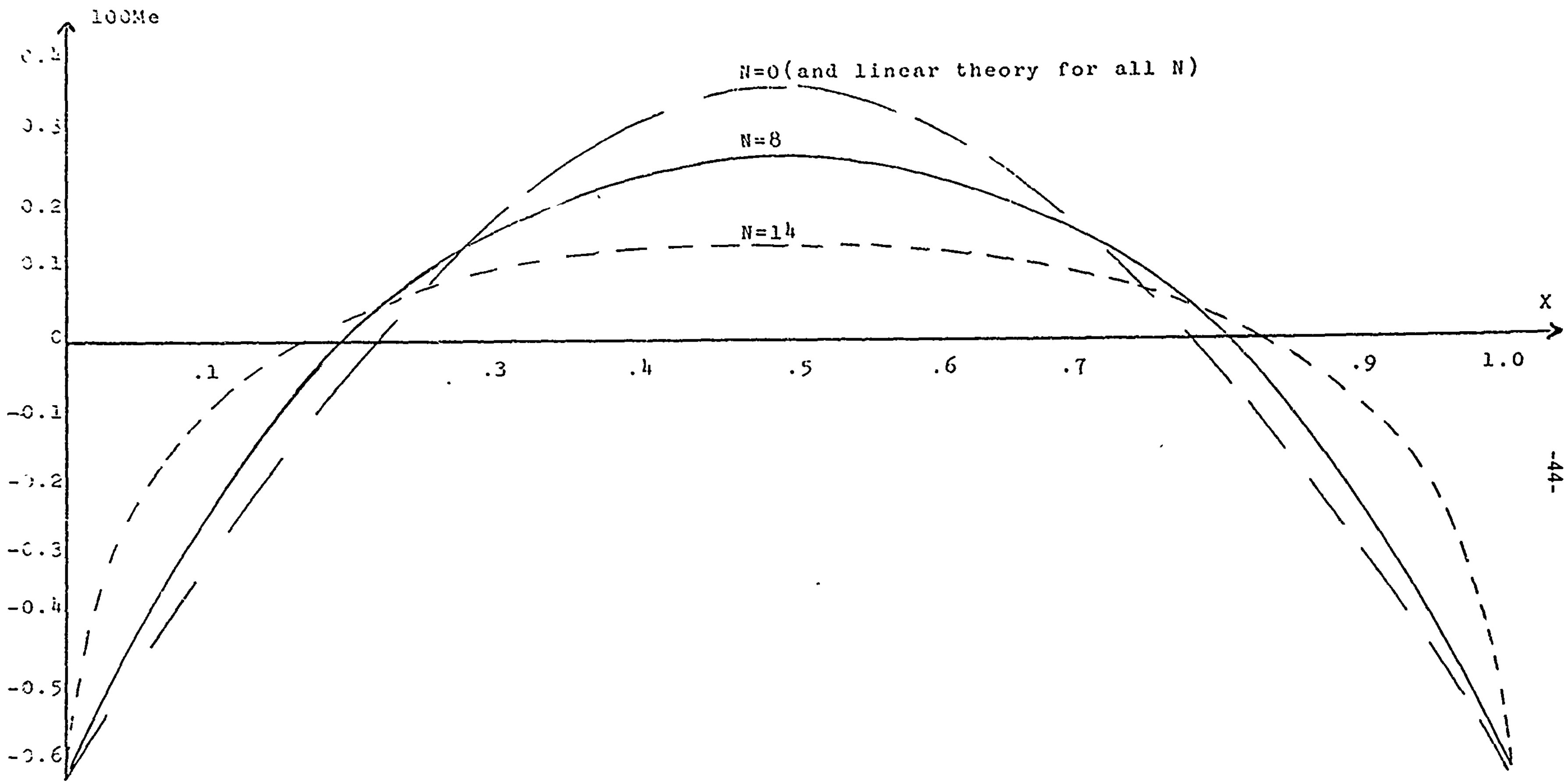

FIGURE 4. Strain versus X at times corresponding to N complete vibrations according to linear theory. At t=0, u=0 and $100Me = \sin \pi X - \frac{2}{\pi}$.

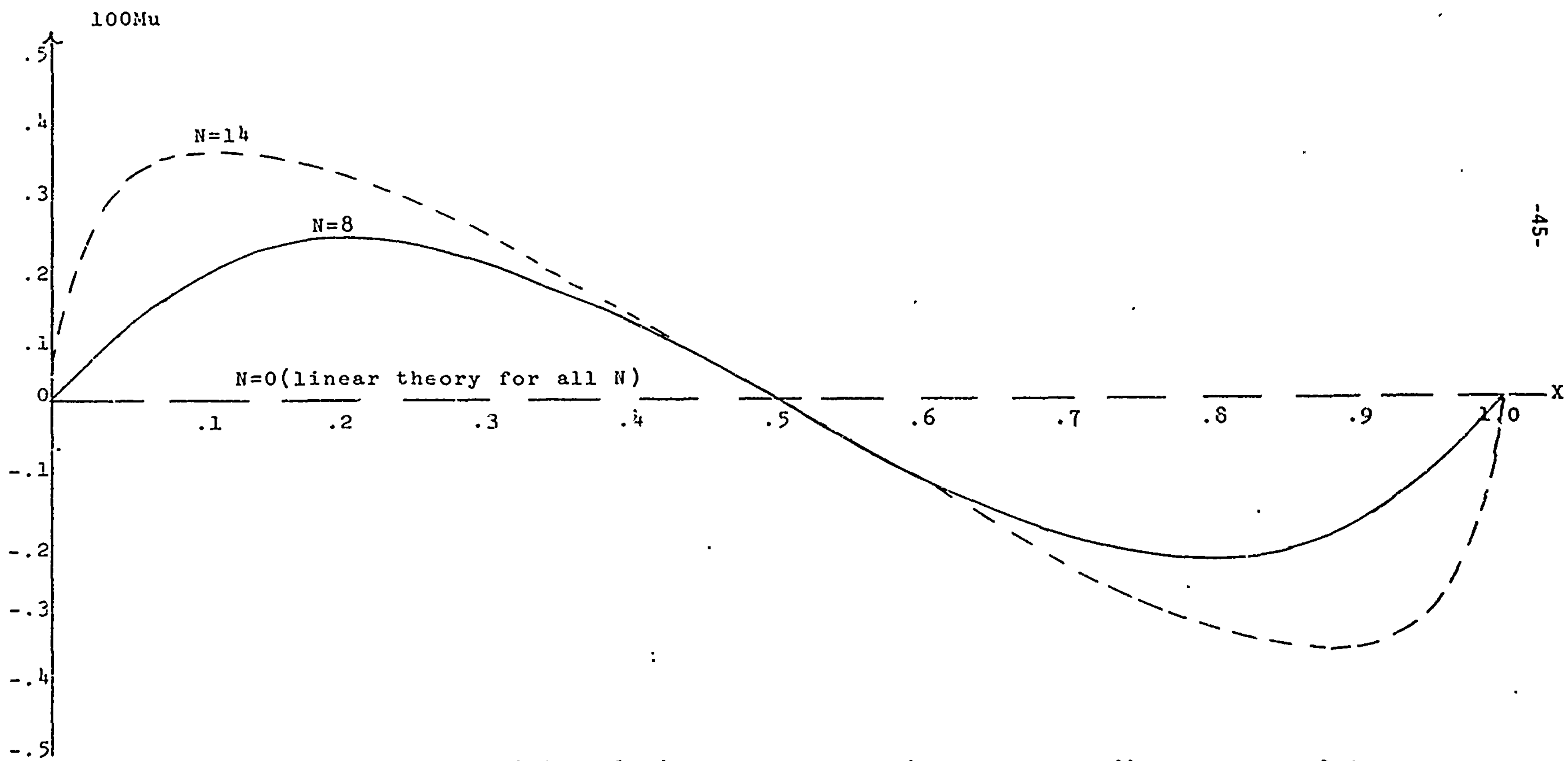

FIGURE 5. Particle velocity versus X at times corresponding to N complete vibrations according to linear theory. At t=0, u≡0 and 100Me = $\sin\pi X - \frac{2}{\pi}$.

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13 ABSTRACT

The equations governing the stretching of a nonlinear elastic panel of finite length are integrated in the limit when the strains are small but the strain rates are comparable with the natural frequency of the panel. It is shown that such a theory must be used when investigating the distortion and decay of a disturbance which is produced by impact and reflection. The mathematical problem involves functions whose values repeat at a sequence of times which are separated by a time interval which depends on the current value of the function.

In a future paper we show that resonant vibrations can also be treated by this theory.

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